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LETTER TO THE EDITOR

A fractal model for band structures just above percolation threshold

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Abstract. A two-dimensional regular-fractal model is proposed to imitate the geometric texture of percolating networks just above percolation threshold. The system is self-similar (fractal) on length scales smaller than the connectedness length, but becomes homogeneous (non-fractal) square lattice on larger length scales. The scaling of vibrational density of states and band structures are studied by using exact renormalisation group methods. It is shown that an infinite number of narrow bands appear to have self-similar structures of typical Cantor sets at low frequency.

Recently, there has been increasing interest in exact mathematical fractals (Mandelbrot 1982, Vicsek 1983, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983). The main reason is that solution of many important equations of physics on these lattices adds to our understanding of the geometric and topological properties that are relevant to modelling the corresponding physical processes. The percolating infinite cluster is one of the most intensively studied random fractals (Deutscher *et al* 1983, Stauffer 1979, Stanley and Coniglio 1983, Kirkpatrick 1979, Kapitulnik and Deutscher 1984). As the concentration p approaches p_c, the pair connectedness length diverges, $\xi \sim (p-p_c)^{-\nu}$. It is generally believed that on large length scales, $L \gg \xi$, the infinite cluster which appears for $p > p_c$ is homogeneous. This homogeneity is believed to disappear for shorter length scales, $L < \xi$. For these scales, the infinite cluster is argued to be self-similar, with a typical fractal dimensionality D (Kapitulnik *et al* 1983, Stanley 1984). Much of the current interest in such systems concentrates on the influence of the geometrical structure on the physical properties in the vicinity of the percolation threshold p_c (Alexander 1983a, Gefen *et al* 1983, 1984, Orbach 1984, Webman 1984).

Various goemetrical models have been proposed to imitate the infinite incipient cluster at the percolation threshold, and it is of great interest to understand the effects of these different geometries on the band structures of elastic vibration at the percolation threshold. Three extreme models for the backbone of the infinite cluster have been proposed, i.e. the family of Sierpinski gaskets, the 'links and nodes' model and the 'links-nodes-blobs' model (Coniglio 1982, Aharony *et al* 1984). Mandelbrot (1984) has also presented a fractal squig model for percolation clusters in the plane to have the geometric and topological properties very close to the infinite cluster. It has been recently proposed that linear physical problems (classical conduction, diffusion and vibration) on percolation clusters at the threshold are governed by the three dimensions d (Euclidean dimension), D (fractal dimension of percolating cluster) and d_s (spectral dimension), and the spectral dimension is $\frac{4}{3}$ independent of the spatial dimension

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(Alexander and Orbach 1982). However, in spite of the interest in the spectral dimension, the band structure does not appear in the literature, except for the Sierpinski gaskets (Alexander 1983b, Domany *et al* 1983, Rammal 1983). Though the percolating network is an ideal system to study the crossover between phonon and fracton, regular models do not appear to mimic the geometric texture and it is not easy to analyse the spectrum (Orbach 1984).

In this letter, we present a two-dimensional regular model for percolating networks just above percolation threshold. We study the scaling of the low-frequency density of states for elastic vibrations and the band structures of allowed regions and gaps in the spectrum. The regular model is self-similar (fractal) on length scales smaller than the connectedness length $\xi_N = 5^N$, but becomes a homogeneous square lattice on large length scales. To mimic the geometric texture of percolating networks just above percolation threshold, our model has the characteristic properties that the infinite cluster is composed of a backbone through which electrical current flows and dangling bonds hanging on it. The backbone consists of multiply connected 'blobs' joined by singly connected 'links'.

The regular model is constructed by the following. Assume first a system of identical masses M placed at the sites of the square lattice, and connected by springs of strength K. Secondly remove masses M placed at the positions (i, j) which satisfy the following relations:

$$
\begin{aligned}
&\cos(2\pi i/5^n + 8\pi/5) = \cos(2\pi j/5^m + 8\pi/5) = 1\\
&\cos(2\pi i/5^n + 6\pi/5) = \cos(2\pi j/5^m + 6\pi/5) = 1,\\
&\cos(2\pi i/5^n + 4\pi/5) = \cos(2\pi j/5^m + 4\pi/5) = 1,\\
&\cos(2\pi i/5^n + 2\pi/5) = \cos(2\pi j/5^m + 2\pi/5) = 1,\\
&\cos(2\pi i/5^n + 2\pi/5) = \cos(2\pi j/5^m + 8\pi/5) = 1,\\
&\cos(2\pi i/5^n + 4\pi/5) = \cos(2\pi j/5^m + 6\pi/5) = 1,\\
&\cos(2\pi i/5^n + 6\pi/5) = \cos(2\pi j/5^m + 4\pi/5) = 1,\\
&\cos(2\pi i/5^n + 8\pi/5) = \cos(2\pi j/5^m + 2\pi/5) = 1 \qquad n, m = 1, 2, \ldots, N.
\end{aligned} \tag{1}
$$

Two construction stages of our regular model are shown by figure 1. The crosses represent the sites where the masses are removed at the first stage, satisfying (1) with $n = m = 1$. The triangular sites represent the masses removed at the second stage, satisfying (1) with $n = m = 2$. The system obtained appears to be a superlattice made by nodes separated by a distance of $\xi_N = 5^N$, connected by quasi-linear links. Within this model, the correlation between two sites at distance $r < \xi_N$ is via a single link, but this link is a branching curve. The curve is identified as one of the branching Koch curves (Gefen *et al* 1983). We obtain the square lattice with self-similar structures on length scales smaller than the connectedness length $\xi_N = 5^N$. The concentration p of masses M is given by

$$p = 1 - [8/25 + 8/(25)^2 + \cdots + 8/(25)^N]. \tag{2}$$

When N is infinitely large, the concentration p approaches the critical value $p_c = \frac{2}{3}$ and the connectedness length diverges, $\xi \sim (p - p_c)^{-1/2}$. Our model with infinitely large N appears to be similar to percolating networks just above percolation threshold.

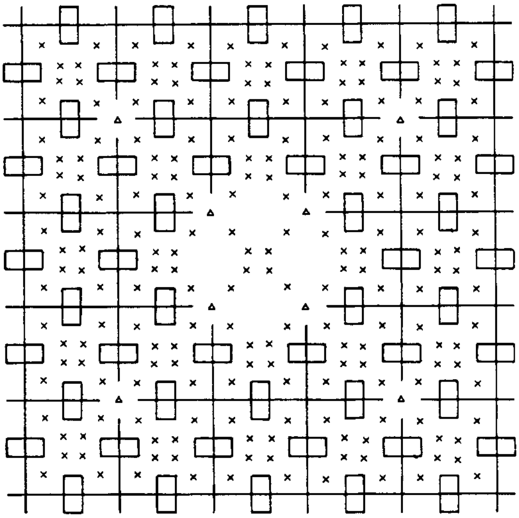

Figure 1. Two construction stages of the regular model. Crosses and triangles denote respectively the sites removed at the first and second stages according to the rule of equations (1).

The quasi-linear fractal lattice is constructed by hierarchical extrapolation. The generator of the fractal lattice is shown in figure 2(*a*). The fractal dimension D is given by $D = \log 23/\log 5$ and corresponds to that of the infinite cluster in our model. The infinite cluster is composed of a backbone through which electrical current flows and dangling bonds hanging on it. The fractal lattice of its backbone in our model is shown by figure 2(*b*). The fractal dimension D_b is given by $D_b = \log 11/\log 5$. The exponent, describing the power-law dependence on scale length L of the conductivity

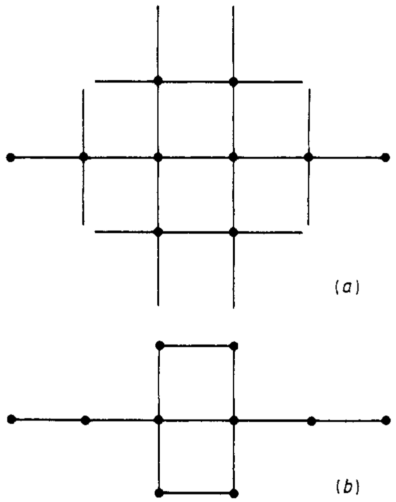

Figure 2. Generators of the fractals for the infinite cluster and its backbone in our regular model. (*a*) and (*b*) correspond respectively to the infinite cluster and its backbone.

$L^{-t/\nu}$, is given by

$$t/\nu = \log R/\log b = \log(23/5)/\log 5 \tag{3}$$

where we define R by assuming that for large n the two-point resistance of an order-n lattice of unit resistors is $\propto R^n$. The value for t/ν is extremely close to that predicted by the $\frac{4}{3}$ hypothesis of Alexander and Orbach.

Here, we compare our regular model with the fractal squig model suggested by Mandelbrot (1984). The fractal squig model possesses the geometric and topological properties very close to the infinite cluster at the percolation threshold but does not describe the approach towards the threshold. Our regular model has the geometric and topological properties similar to the infinite cluster and can describe the approach towards the threshold according as N increases. One may study the crossover between phonon and fracton by the regular model.

We study elastic vibration on the regular fractal model. Let $\alpha = M\omega^2/K = \omega^2/\omega_0^2$ denote the reduced squared frequency and $\{U_j\, e^{i\omega t}\}$ the eigenstate associated with a mode of frequency ω. The set of equations of motion for sites i is given by

$$\alpha U_i = \sum_j (U_i - U_j) \tag{4}$$

where j denotes a neighbouring site of i.

The self-similarity of the lattice on length scales smaller than the connectedness length $\xi_N = 5^N$ leads to a natural decimation procedure (Kadanoff and Houghton 1975, Goncalves da Silva and Koiller 1981). The free boundary conditions at the edges of the dangling bonds introduce a complication, because three types of sites are not equivalent with different numbers of dangling bonds. Three parameters are involved in the decimation procedure because of different types of renormalised sites. The idea involves eliminating the lowest scale amplitudes in equation (4). This procedure leads to a reduced set of equations describing the same physics on a lattice scaled down by a factor $b = 5$. This exact renormalisation leads to three renormalised frequencies (three parameter renormalisation group). In the following we shall illustrate this procedure. The decimation technique is schematically indicated in figure 3. The sites

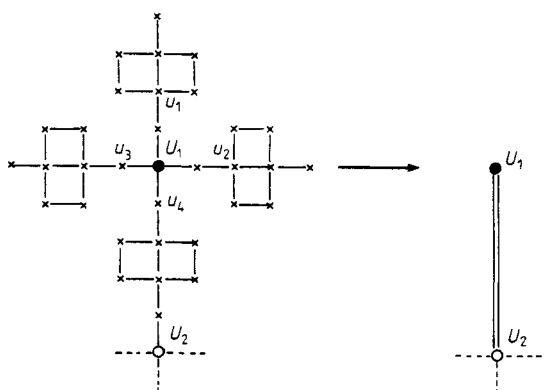

Figure 3. Schematic representation of the decimation technique for a part of the fractal lattice. The sites denoted by crosses on the left-hand lattice are eliminated, producing the right-hand renormalised lattice.

represented by crosses on the left-hand lattice are eliminated, producing the renormalised right-hand lattice. We classify the renormalised sites into three types of sites: (a) the sites with three dangling bonds, (b) the sites with two dangling bonds, and (c) the sites with no dangling bonds (see figure 4). We shall consider the equation of motion for the sites of type a (see figure 3):

$$\alpha_c U_1 = 4U_1 - u_1 - u_2 - u_3 - u_4. \tag{5}$$

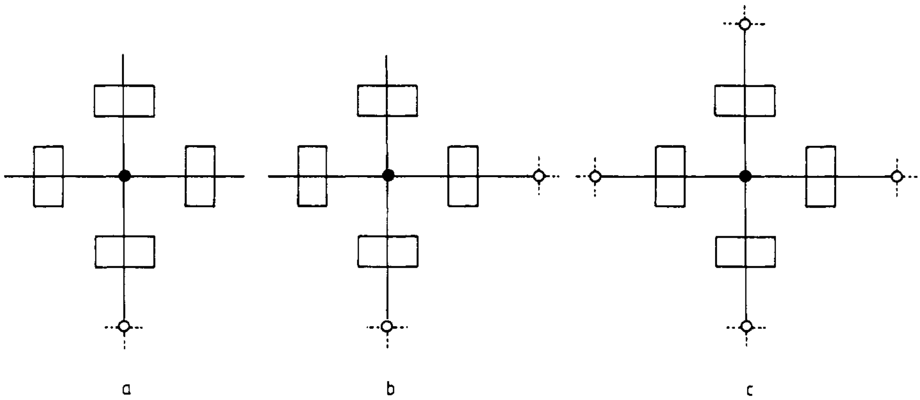

Figure 4. Classification of the renormalised sites which are indicated by full circles. (a) The sites with three dangling bonds, (b) the sites with two dangling bonds, and (c) the sites with no dangling bonds.

Eliminating $\{u_i\}$, one obtains a new equation for $\{U_i\}$:

$$\alpha'_a U_1 = U_1 - U_2. \tag{6}$$

This relation can be cast in the form of the first equations with the renormalisation:

$$\alpha'_a = (\alpha_c - 4)\det C/\det E - \det D/\det E - 3(\det B \det C)/(\det A \det E) + 1. \tag{7}$$

Similarly, one can obtain the recursion relations of renormalised frequencies for sites of types b and c:

$$\alpha'_b = (\alpha_c - 4)\det C/\det E - 2\det D/\det E - 2(\det B \det C)/(\det A \det E) + 2, \tag{8}$$

$$\alpha'_c = (\alpha_c - 4)\det C/\det E - 4\det D/\det E + 4. \tag{9}$$

At low frequency, the recursion relations (7), (8) and (9) yield the common fixed point $\alpha^* = 0$, implying the existence of a uniform mode at zero frequency. In the vicinity of this fixed point, we have

$$\begin{pmatrix} \delta\alpha'_b \\ \delta\alpha'_b \\ \delta\alpha'_c \end{pmatrix} = \begin{pmatrix} \frac{69}{5}, \frac{414}{5}, \frac{184}{5} \\ \frac{46}{5}, \frac{368}{5}, \frac{161}{5} \\ 0, \frac{276}{5}, 23 \end{pmatrix} \begin{pmatrix} \delta\alpha_a \\ \delta\alpha_b \\ \delta\alpha_c \end{pmatrix}. \tag{10}$$

The matrix in equation (10) has eigenvalues $\lambda = 0, \frac{23}{5}, \frac{529}{5}$. Following the scaling method by Rammal and Toulouse (1983), the spectral dimension is given

$$d_s = 2D \log b/\log \lambda_{max} = 2\log 23/\log \tfrac{529}{5}. \tag{11}$$

This result agrees with the value derived by assuming Einstein's relation: $d_s = 2D/(2+t/\nu - d + D)$. Table 1 lists the geometric and physical properties, determined analytically by our regular-fractal model.

Table 1. List of the geometric and physical properties determined analytically by our regular model.

p_c	ν	D	D_b	t/ν	d_s
$\frac{2}{3}$	$\frac{1}{2}$	$\log 23/\log 5$	$\log 11/\log 5$	$\log \frac{23}{5}/\log 5$	$2\log 23/\log \frac{529}{5}$

On the other hand, the regular model becomes a homogeneous square lattice on large length scales, $L \gg \xi$. The density of vibrational states in the regime can be written as

$$\rho(\omega) \sim \omega^{d-1}. \tag{12}$$

We study the band structure of allowed regions and gaps of spectrum to examine whether or not the crossover between the long wavelength phonon and the short-length-scale fracton vibrational excitations happen in our regular model. The regular model is self-similar (fractal) on length scales smaller than the connectedness length $\xi_N = 5^N$, but becomes a homogeneous square lattice on large length scales. The renormalised lattice obtained after the Nth renormalisation appears to be the ordered square lattice on which elastic vibrations are governed by the equations of motion with the renormalised frequency α_c^N. Allowed regions of the spectrum are therefore given by

$$0 \leqslant \alpha_c^N \leqslant 8, \tag{13}$$

where α_c^N indicates the renormalised frequency obtained after the Nth substitution of equations (7), (8) and (9) into themselves. Figure 5 shows allowed regions of the spectrum as a function N, obtained by numerical calculations of the relation (13) with

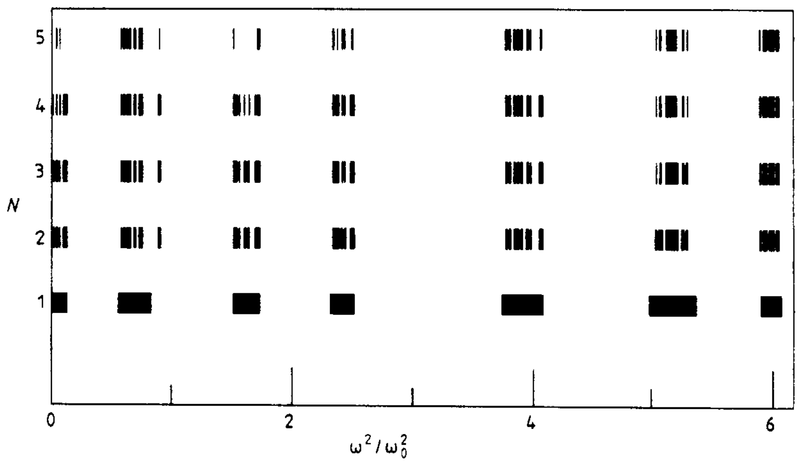

Figure 5. Allowed frequency regions in our regular model. The band structures are shown for $N = 1, 2, \ldots, 5$.

N times iterations of recursion relations (7), (8) and (9). Figure 6, a magnification of the part of figure 5 near the zero frequency, shows the allowed frequency regions. They are obtained by successively expanding the frequency scale by a factor $\frac{529}{5}$. In spite of increasing magnification of the frequency scale in the vicinity of the zero frequency, similar structures appear which reveal a self-similarity to be a typical Cantor set. As $N \to \infty$, what we see is an infinite number of very narrow bands which have self-similar structures. We rederive the scaling form for the frequency in the vicinity of the zero frequency. Notice that, for large N, the allowed regions get narrower and narrower. The continuous spectrum with the very narrow bandwidth at zero frequency corresponds to the long-wavelength phonon modes. This bandwidth scales as

$$\Delta\omega_N \sim (\lambda_{\max})^{N/2} = \xi_N^{\ln\lambda_{\max}/2\ln 5} = \xi_N^{-(2+\theta)/2}. \tag{14}$$

This corresponds to the crossover frequency ω_{c0}. The band structure in our model shows phonon modes for $\omega < \omega_{co}$ and fraction modes for $\omega \gg \omega_{co}$.

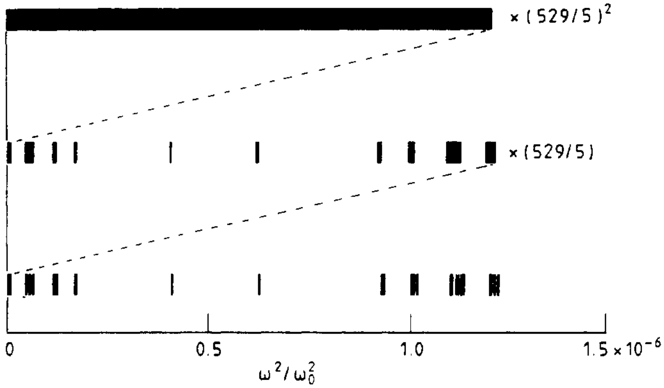

Figure 6. The allowed frequency regions. A magnification of the part of figure 5 near the zero frequency ($N = 5$). They are respectively obtained by successively expanding the reduced squared frequency scale by a factor $\frac{529}{5}$.

In conclusion, we summarise that microscopically self-similar (fractal) structures lead to the scaling of the low-frequency density of states via recursion relations (10) and on the other hand the property of the macroscopically homogeneous square lattice leads to the band structures of the allowed frequency regions via relation (13) with the recursion relations (7), (8) and (9).

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